

# Weak operator topology, operator ranges and operator equations via Kolmogorov widths

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February 19, 2009

**Abstract:** Let  $K$  be an absolutely convex infinite-dimensional compact in a Banach space  $\mathcal{X}$ . The set of all bounded linear operators  $T$  on  $\mathcal{X}$  satisfying  $TK \supset K$  is denoted by  $G(K)$ . Our starting point is the study of the closure  $WG(K)$  of  $G(K)$  in the weak operator topology. We prove that  $WG(K)$  contains the algebra of all operators leaving  $\overline{\text{lin}(K)}$  invariant. More precise results are obtained in terms of the Kolmogorov  $n$ -widths of the compact  $K$ . The obtained results are used in the study of operator ranges and operator equations.

**Mathematics Subject Classification:** Primary 47A05; Secondary 41A46, 47A30, 47A62.

**Keywords:** Banach space; bounded linear operator; Hilbert space; Kolmogorov width; operator equation; operator range; strong operator topology; weak operator topology

## 1 Introduction

Let  $K$  be a subset in a Banach space  $\mathcal{X}$ . We say (with some abuse of the language) that an operator  $D \in \mathcal{L}(\mathcal{X})$  covers  $K$ , if  $DK \supset K$ . The set of all operators covering  $K$  will be denoted by  $G(K)$ . It is a semigroup with a unit since the identity operator is in  $G(K)$ . It is easy to check that if  $K$  is compact then  $G(K)$  is closed in the norm topology and, moreover, sequentially closed in the weak operator topology (WOT). It is somewhat surprising that for each absolutely convex infinite dimensional compact  $K$  the WOT-closure of  $G(K)$  is much larger than  $G(K)$  itself, and in many cases it coincides with the algebra  $\mathcal{L}(\mathcal{X})$  of all operators on  $\mathcal{X}$ . Our aim is to understand: how much

freedom has an operator which is obliged to cover a given compact? In a simplest form the question is: “How large is  $G(K)$ ?”. We answer this question describing the WOT-closure  $WG(K)$  of  $G(K)$  as well as its closure in the ultra-weak topology (for the case of Hilbert spaces). These results are obtained in Sections 2–3 for the Banach spaces, and in more detailed form in Section 4 for Hilbert spaces; they are formulated in terms of Kolmogorov’s  $n$ -widths of  $K$ .

In Section 5 we consider a more general object: the set  $G(K_1, K_2)$  of all operators  $T$  which have the property  $TK_1 \supset K_2$  where  $K_1, K_2$  are fixed convex compacts in Hilbert spaces.

In further sections we apply the obtained results for study of some related subjects: operator ranges (Section 7), operator equations of the form  $XAY = B$  (Section 8) and operators with the property

$$\|AXx\| \geq \|Ax\| \text{ for all } x \in \mathcal{H}$$

where  $A$  is a given operator on a Hilbert space  $\mathcal{H}$ . Some applications of the obtained results to the theory of quadratic operator inequalities and operator fractional linear relations will be presented in a subsequent work. In fact our interest to the semigroups  $G(K)$  was initially motivated by these applications; the relations to other topics became clear for us in the process of the study.

**Notation.** Our terminology and notation of Banach space theory follows [10]. Our definitions of the standard topologies on spaces of operators follow [3, Chapter VI]. Let  $\mathcal{X}, \mathcal{Y}$  be Banach spaces. We denote the closed unit ball of a Banach space  $\mathcal{Y}$  by  $B_{\mathcal{Y}}$ , and the norm closure of a set  $M \subset \mathcal{Y}$  by  $\overline{M}$ . We denote the set of bounded linear operators from  $\mathcal{Y}$  to  $\mathcal{X}$  by  $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ . We write  $\mathcal{L}(\mathcal{X})$  for  $\mathcal{L}(\mathcal{X}, \mathcal{X})$ . The identity operator in  $\mathcal{L}(\mathcal{X})$  will be denoted by  $I$ .

Throughout the paper we denote by  $\text{lin}(K)$  the linear span of a set  $K$ , and by  $V_K$  the closed subspace spanned by  $K$ , that is,  $V_K = \overline{\text{lin}(K)}$ . We denote by  $\mathcal{A}_K$  the algebra of all operators for which  $V_K$  is an invariant subspace. It is clear that  $\mathcal{A}_K$  is closed in the WOT.

**Remark on related work.** Coverings of compacts by sets of the form  $R(B_{\mathcal{Z}})$  where  $\mathcal{Z}$  is a Banach space and  $R \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$  have been studied by many authors, see [1], [2], and [6]. However, the main foci of these papers are different. In all of the mentioned papers additional conditions are imposed on  $\mathcal{Z}$ , or on  $R$ , or on both of them, and the main problem is: whether such  $R$  exist? In the context of the present paper existence is immediate, while for us (as it was mentioned above) the main question is: “How large is the set of such operators?”.

**Acknowledgement.** The authors would like to thank Heydar Radjavi for a helpful discussion and his interest in our work.

We finish the introduction by showing that for non-convex compacts  $K$  the semigroup  $G(K)$  can be trivial:

**Example 1.1.** *There exists a compact  $K$  in an infinite dimensional separable Hilbert space  $\mathcal{H}$  such that the only element of  $G(K)$  is the identity operator.*

*Proof.* Let  $\{e_n\}_{n=1}^{\infty}$  be an orthonormal basis in  $\mathcal{H}$ ;  $\{\alpha_n\}_{n=1}^{\infty}$  be a sequence of real numbers satisfying  $\alpha_n > 0$  and  $\lim_{n \rightarrow \infty} \alpha_{n+1}/\alpha_n = 0$ ; and  $\{\beta_n\}_{n=1}^{\infty}$  be a sequence of distinct numbers in the open interval  $(\frac{1}{2}, 1)$ . The compact  $K$  is defined by

$$K = \{0\} \cup \{\alpha_n e_n\}_{n=1}^{\infty} \cup \{\alpha_n \beta_n e_n\}_{n=1}^{\infty}.$$

Assume that there exists  $D \in \mathcal{L}(\mathcal{H})$  such that  $D(K) \supset K$  and  $D$  is not the identity operator. Let  $M = \{n \in \mathbb{N} : De_n \neq e_n\}$ . Since  $\{e_n\}_{n=1}^\infty$  is a basis in  $\mathcal{H}$  and  $D$  is not the identity operator, the set  $M$  is nonempty. We introduce the following oriented graph with the vertex set  $M$ . There is an oriented edge  $\overrightarrow{nm}$  starting at  $n \in M$  and ending at  $m \in M$  ( $n$  can be equal to  $m$ ) if and only if one of the following equalities holds:

$$\begin{aligned} D(\alpha_m e_m) &= \alpha_n e_n, \quad D(\beta_m \alpha_m e_m) = \alpha_n e_n, \\ D(\alpha_m e_m) &= \beta_n \alpha_n e_n, \quad D(\beta_m \alpha_m e_m) = \beta_n \alpha_n e_n. \end{aligned} \tag{1}$$

**Important observation.** Since the numbers  $\{\beta_n\}_{n=1}^\infty \subset (\frac{1}{2}, 1)$  are distinct, the number of edges starting at  $n$  is at least 2 for each  $n \in M$ , while there is at most one edge ending at  $m \in M$ .

An immediate consequence of this observation is that there are infinitely many oriented edges  $\overrightarrow{nm}$  with  $n < m$ , that is, infinitely many pairs  $(n, m)$ ,  $n < m$ , for which one of equalities from (1) holds. Taking into account the conditions satisfied by  $\{\alpha_n\}$  and  $\{\beta_n\}$ , we get a contradiction with the boundedness of  $D$ .  $\square$

This example shows that in the general case there is a very strong dependence of the size of the semigroup  $G(K)$  on the geometry of  $K$ . To relax this dependence we restrict our attention to absolutely convex compacts  $K$ .

## 2 $\mathcal{A}_K \subset WG(K)$

**Theorem 2.1.** *Let  $K$  be an absolutely convex infinite-dimensional compact. Then  $\mathcal{A}_K \subset WG(K)$ .*

**Remark 2.2.** *Theorem 2.1 is no longer true for finite dimensional compacts. In fact, if  $K$  is absolutely convex finite-dimensional compact, then  $A \in G(K)$  implies that  $A$  leaves  $V_K$  invariant. Since  $V_K$  is finite dimensional, the condition  $AK \supset K$  passes to operators from the WOT-closure. Thus  $WG(K)$  is a proper subset of  $\mathcal{A}_K$ .*

Let  $F$  be a subset of  $\mathcal{X}^*$ . We use the notation  $F_\perp$  for the *pre-annihilator* of  $F$ , that is,  $F_\perp := \{x \in \mathcal{X} : \forall f \in F \ f(x) = 0\}$ .

**Lemma 2.3.** *Let  $K$  be an absolutely convex infinite-dimensional compact in a Banach space  $\mathcal{X}$ . For each finite dimensional subspace  $F \subset \mathcal{X}^*$ , each finite dimensional subspace  $\mathcal{Y} \subset \mathcal{X}$ , and an arbitrary linear mapping  $N : \mathcal{Y} \rightarrow \mathcal{X}$  satisfying  $N(\mathcal{Y} \cap V_K) \subset \text{lin}(K)$ , there is  $D \in G(K)$  satisfying the condition*

$$Dx - Nx \in F_\perp \quad \forall x \in \mathcal{Y}. \tag{2}$$

*Proof.* Note first of all that it suffices to prove the lemma under an additional assumption that  $\mathcal{Y} \subset V_K$ . Indeed, suppose that it is done, then in the general case we choose a complement  $\mathcal{Y}_1$  of  $\mathcal{Y} \cap V_K$  in  $\mathcal{Y}$  and choose a complement  $\mathcal{X}_1$  of  $\mathcal{Y}_1$  in  $\mathcal{X}$  that contains  $V_K$ . By our assumption there is an operator  $D \in \mathcal{L}(\mathcal{X}_1)$  with  $DK \supset K$  and  $Dx - Nx \in F_\perp \cap \mathcal{X}_1 \ \forall x \in \mathcal{Y} \cap V_K$ . It remains to extend  $D$  to  $\mathcal{X}$  setting  $Dx = Nx$  for  $x \in \mathcal{Y}_1$ .

So we assume that  $\mathcal{Y} \subset V_K$ . For brevity denote  $\text{lin}(K)$  by  $\mathcal{Z}$ .

Let  $P : \mathcal{X} \rightarrow \mathcal{X}$  be a projection of finite rank, such that  $P\mathcal{X} \supset \mathcal{Y} + N\mathcal{Y}$ ,  $(I - P)\mathcal{X} \subset F_\perp$  and  $\dim(P\mathcal{Z}) \geq \dim F + \dim \mathcal{Y}$ . The last condition can be satisfied since  $K$  is finite-dimensional.

The conditions  $N(\mathcal{Y} \cap V_K) \subset \text{lin}(K)$  and  $\mathcal{Y} \subset V_K$  imply that the subspace  $N\mathcal{Y}$  is contained in  $\mathcal{Z} \cap P\mathcal{X}$ . The space  $\ker(P) \cap \mathcal{Z}$  has finite codimension in  $\mathcal{Z}$ . Therefore there exists a complement  $L$  of  $\ker(P) \cap \mathcal{Z}$  in  $\mathcal{Z}$  such that  $L \supset N\mathcal{Y}$ .

We have  $PL = P\mathcal{Z}$  and  $L \cap (I - P)\mathcal{X} = \{0\}$ . Since the subspace  $(I - P)\mathcal{X}$  has finite codimension in  $\mathcal{X}$ , we can find a subspace  $M \supset L$ , which is a complement of  $(I - P)\mathcal{X}$  in  $\mathcal{X}$ . Let  $Q_M : \mathcal{X} \rightarrow M$  be the projection onto  $M$  with the kernel  $\ker(P) = (I - P)\mathcal{X}$ , and let  $M_0$  be the complement of  $L$  in  $M$ . Since  $\mathcal{Z} = (\ker P \cap \mathcal{Z}) \oplus L$  and  $Q_M(L) = L$ , we have  $Q_M(\mathcal{Z}) = L \subset \mathcal{Z}$ .

To introduce an operator  $D \in \mathcal{L}(\mathcal{X})$  it suffices to determine its action on  $\ker P$  and on  $M$ . We do it in the following way:

(a) The restriction of  $D$  to  $\ker P$  is a multiple  $\lambda I_{\ker P}$  of the identity operator, where  $\lambda$  is chosen in such a way that  $(I - Q_M)K \subset \frac{\lambda}{2}K$  (such a choice is possible because  $(I - Q_M)K$  is a compact subset of  $\mathcal{Z} = \cup_{n \in \mathbb{N}} nK$ ).

(b) The restriction of  $D$  to  $M$  is defined in three ‘pieces’:

- $D|_{M_0} = 0$ .
- Now we define the restriction of  $D$  to  $Q_M(\mathcal{Y})$ . Observe that  $Q_M(\mathcal{Y}) \subset L$ . This follows from

$$\mathcal{Y} \subset V_K \subset L \oplus (I - P)\mathcal{X}.$$

In addition  $Q_M|_{\mathcal{Y}}$  is an isomorphism, because  $\mathcal{Y} \subset P\mathcal{X}$ , and  $P\mathcal{X}$  and  $M$  are complements of the same subspace. Because of this, the operator  $D|_{Q_M(\mathcal{Y})}$  given by

$$D(Q_M(x)) = Nx + \alpha S(Q_M(x)) \text{ for } x \in \mathcal{Y}$$

is well-defined, where  $\alpha \in \mathbb{R}$  and  $S$  is an isomorphism of  $Q_M(\mathcal{Y})$  into  $F_{\perp} \cap L$ . Such isomorphisms exist because  $\dim L \geq \dim(PL) = \dim(P\mathcal{Z})$ , and we assumed that  $\dim(P\mathcal{Z}) \geq \dim F + \dim \mathcal{Y}$ . Now we choose  $\alpha$  to be so large that the image of  $K \cap Q_M(\mathcal{Y})$  covers a ‘large’ multiple of the intersection of  $Q_M(K)$  with the space onto which it maps. This is possible because zero has non-empty interior in  $K \cap Q_M(\mathcal{Y})$  and  $Q_M(K)$  is compact.

- We define  $D$  on the complement of  $Q_M(\mathcal{Y})$  in  $L$  as a ‘dilation’ operator onto some complement of the  $D(Q_M(\mathcal{Y}))$  in  $L$ . The number  $\alpha$  and the dilation are selected in such a way that

$$D(K \cap L) \supset 2Q_M(K). \quad (3)$$

To see that it is possible recall that  $Q_M(K) \subset L$  and, since  $L$  is finite-dimensional, the set  $K \cap L$  contains a multiple of the unit ball of  $L$ .

It remains to verify that  $D$  satisfies the conditions (2) and  $DK \supset K$ .

Condition (2). Let  $x \in \mathcal{Y}$ , then  $x = Q_M x + (I - Q_M)x$ . Therefore

$$Dx = Nx + \alpha S(Q_M(x)) + \lambda(I - Q_M)x.$$

Let  $f \in F$ . We get:

$$f(Dx) = f(Nx) + \alpha f(S(Q_M(x))) + \lambda f(I - Q_M)x = f(Nx),$$

where we use the following facts: (a) The image of  $S$  is in  $F_{\perp}$ ; (b)  $(I - Q_M)\mathcal{X} = (I - P)\mathcal{X} \subset F_{\perp}$ .

Condition  $DK \supset K$ . Let  $x \in K$ . Then  $x = Q_M x + (I - Q_M)x$ . The condition (3) implies that there exists  $v \in \frac{1}{2}(L \cap K)$  such that  $Dv = Q_M x$ . The choice of  $\lambda$  implies that

$w = \frac{1}{\lambda}(I - Q_M)x$  satisfies  $w \in \frac{1}{2}K$ . Let  $z = v + w$ . It is clear that  $z \in K$ . We need to show that  $Dz = x$ . We have

$$\begin{aligned} Dz &= Dv + Dw = Dv + D\left(\frac{1}{\lambda}(I - Q_M)x\right) \\ &= Q_Mx + \lambda\left(\frac{1}{\lambda}(I - Q_M)x\right) = x. \end{aligned}$$

(We use the fact that  $(I - Q_M)\mathcal{X} \subset \ker P$ .) □

*Proof of Theorem 2.1.* Let  $T \in \mathcal{A}_K$ , and

$$\mathcal{U} = \{E \in \mathcal{L}(\mathcal{X}) : \forall i \in \{1, \dots, n\} \ |f_i(Ex_i)| < \varepsilon\}$$

be a WOT-neighborhood of 0 in  $\mathcal{L}(\mathcal{X})$ , where  $n \in \mathbb{N}$ ,  $\varepsilon > 0$ ,  $\{f_i\}_{i=1}^n \in \mathcal{X}^*$  and  $\{x_i\}_{i=1}^n \in \mathcal{X}$ . We need to show that  $T + \mathcal{U}$  contains an operator from  $G(K)$  for each choice of  $n, \varepsilon, f_i$ , and  $x_i$ . Let  $F = \text{lin}(\{f_i\}_{i=1}^n)$  and  $\mathcal{Y} = \text{lin}(\{x_i\}_{i=1}^n)$ . Let  $\mathcal{Y}_1 = \mathcal{Y} \cap V_K$ . Since  $T \in \mathcal{A}_K$ , we have  $T(\mathcal{Y}_1) \subset V_K$ . Since  $V_K = \overline{\mathcal{Z}}$ , we can find a “slight perturbation”  $\tilde{T}$  of  $T$  satisfying  $\tilde{T}(\mathcal{Y}_1) \subset \mathcal{Z}$ . In particular, we can find such  $\tilde{T}$  in  $T + \frac{1}{2}\mathcal{U}$ . It remains to show that  $\tilde{T} + \frac{1}{2}\mathcal{U}$  contains an operator  $D$  from  $G(K)$ .

It is clear that each operator  $S$  satisfying

$$\forall x \in \mathcal{Y} \quad Sx - \tilde{T}x \in F_\perp$$

is in  $\tilde{T} + \frac{1}{2}\mathcal{U}$ . Now the existence of the desired operator  $D$  is an immediate consequence of Lemma 2.3 applied to  $N = \tilde{T}$ . □

**Corollary 2.4.** *If  $V_K = \mathcal{X}$ , then  $WG(K) = \mathcal{L}(\mathcal{X})$ .*

### 3 Application of Kolmogorov $n$ -widths to estimates of the ‘size’ of $WG(K)$ from above

We are going to use the notion of Kolmogorov  $n$ -width. In this respect we follow the terminology and notation of the book [17, Chapter II]. Let  $\mathcal{Z}$  be a subset of a Banach space  $\mathcal{X}$  and  $x \in \mathcal{X}$ . The *distance from  $x$  to  $\mathcal{Z}$*  is defined as

$$E(x, \mathcal{Z}) = \inf\{\|x - z\| : z \in \mathcal{Z}\}.$$

**Definition 3.1.** Let  $K$  be a subset of a Banach space  $\mathcal{X}$ ,  $n \in \mathbb{N} \cup \{0\}$ . The *Kolmogorov  $n$ -width* of  $K$  is given by

$$d_n(K) = \inf_{\mathcal{X}_n} \sup_{x \in K} E(x, \mathcal{X}_n),$$

where the infimum is over all  $n$ -dimensional subspaces. If

$$d_n(K) = \sup_{x \in K} E(x, \mathcal{Z})$$

and  $\mathcal{Z} \subset \mathcal{X}$  is an  $n$ -dimensional subspace, then  $\mathcal{Z}$  is said to be an *optimal* subspace for  $d_n(K)$ .

**Lemma 3.2.** *Let  $K$  and  $K_0$  be two subsets in a Banach space  $\mathcal{X}$  and  $D \in \mathcal{L}(\mathcal{X})$  be such that  $D(K_0) \supset K$ . Then  $d_n(K) \leq \|D\|d_n(K_0)$  for all  $n \in \mathbb{N} \cup \{0\}$ .*

*Proof.* Let  $\mathcal{Z} \subset \mathcal{X}$  be an  $n$ -dimensional subspace. Then  $D\mathcal{Z} \subset \mathcal{X}$  is a subspace of dimension  $\leq n$  and  $E(Dx, D\mathcal{Z}) \leq \|D\|E(x, \mathcal{Z})$ . The conclusion follows.  $\square$

**Lemma 3.3.** *Let  $K$  be a bounded subset in a Banach space  $\mathcal{X}$ . If  $K_0 = K \cap L$ , where  $L$  is a closed linear subspace in  $\mathcal{X}$  which does not contain  $K$ , then there exists a constant  $0 < C < \infty$  such that  $d_n(K_0) \leq Cd_{n+1}(K)$  for all  $n \in \mathbb{N} \cup \{0\}$ .*

*Proof.* It is well-known (see [17, p. 10]) that a bounded set  $K$  is compact if and only if  $\lim_{n \rightarrow \infty} d_n(K) = 0$ . Therefore it suffices to consider the case when  $K$  is compact. It is clearly enough to consider the case when  $L$  is a subspace of codimension 1. Let  $L = \ker \nu$  where  $\nu \in \mathcal{X}^*$ . We may assume without loss of generality that the norm of the restriction of  $\nu$  to  $\text{lin}K$  satisfies  $\|\nu|_{\text{lin}(K)}\| = 1$ . For each  $n \in \mathbb{N} \cup \{0\}$  let  $L_n \subset \mathcal{X}$  be a subspace of dimension  $n$  satisfying  $\sup_{x \in K} E(x, L_n) \leq 2d_n(K)$ .

First we show that there exists  $N \in \mathbb{N}$  such that  $\|\nu|_{L_M}\| > \frac{1}{2}$  for all  $M \geq N$ . Let  $0 < \varepsilon < 1$  and let  $x_i \in K$  and scalars  $a_i$  ( $i = 1, \dots, k$ ) be such that the vector  $h = \sum_i a_i x_i$  satisfies  $\|h\| = 1$  and  $\nu(h) > 1 - \varepsilon$ . Let  $\delta > 0$  be such that  $\delta\|\nu\| \sum |a_i| < \varepsilon$ . Let  $N$  be such that for  $M \geq N$  we have  $d_M(K) < \delta/2$ . Then for  $M \geq N$  there exist  $y_i \in L_M$  such that  $\|x_i - y_i\| < \delta$ . Therefore the vector  $g := \sum_i a_i y_i$  satisfies  $\|\nu\| \cdot \|g - h\| < \varepsilon$  and  $g \in L_M$ . Choosing appropriate  $\varepsilon$  and  $\delta$  we get  $\|\nu|_{L_M}\| > \frac{1}{2}$ .

Let  $M \geq N$  and let  $L_{M,0} = L_M \cap \ker \nu$ . Let  $x \in K_0$ . We are going to show that  $E(x, L_{M,0}) < (2\|\nu\| + 1)2d_M(K)$ . By the definition of  $L_M$  there is  $y \in L_M$  such that  $\|x - y\| \leq 2d_M(K)$ . Since  $\nu(x) = 0$ , we have  $|\nu(y)| \leq \|\nu\|2d_M(K)$ . Since  $\|\nu|_{L_M}\| > \frac{1}{2}$ , we conclude that  $E(y, L_{M,0}) < 4\|\nu\|d_M(K)$ . Therefore  $E(x, L_{M,0}) < (2\|\nu\| + 1)2d_M(K)$ . It is clear that  $\dim L_{M,0} = M - 1$ . Thus for  $M \geq N$  we have  $d_{M-1}(K_0) < (2\|\nu\| + 1)2d_M(K)$ . The conclusion follows.  $\square$

**Definition 3.4.** Let  $\{a_n\}$  be a non-increasing sequence of positive numbers satisfying  $\lim_{n \rightarrow \infty} a_n = 0$ . We say that  $\{a_n\}$  is *lacunary* if

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0. \quad (4)$$

**Lemma 3.5.** *If the sequence  $\{d_n(K)\}_{n=1}^\infty$  is lacunary, then  $G(K) \subset \mathcal{A}_K$ .*

*Proof.* Let  $R \in \mathcal{L}(\mathcal{X})$  be such that  $RV_K$  is not contained in  $V_K$ . We have to show that  $RK$  does not contain  $K$ . Assume the contrary.

It follows from our assumption that  $R^{-1}(V_K)$  is a proper subspace of  $V_K$  and  $R(K_0) \supset K$  where  $K_0 = K \cap R^{-1}(V_K)$  is a proper section of  $K$ .

By Lemma 3.2 we get  $d_n(K) \leq \|R\|d_n(K_0)$  for all  $n \in \mathbb{N} \cup \{0\}$ . By Lemma 3.3 we get  $d_n(K_0) \leq Cd_{n+1}(K)$  for some  $0 < C < \infty$  (which depends on  $K$  and  $K_0$ , but not on  $n$ ) and all  $n \in \mathbb{N} \cup \{0\}$ . We get  $d_{n+1}(K) \geq (C\|R\|)^{-1}d_n(K)$ , hence the sequence  $\{d_n(K)\}_{n=1}^\infty$  is not lacunary. We get a contradiction.  $\square$

**Remark 3.6.** *The assumptions of convexity and symmetry of  $K$  are not needed in Lemmas 3.2, 3.3, and 3.5.*

Combining Theorem 2.1 and Lemma 3.5 we get

**Theorem 3.7.** *If an absolutely convex compact  $K$  is such that the sequence  $\{d_n(K)\}_{n=1}^\infty$  is lacunary, then  $WG(K) = \mathcal{A}_K$ .*

## 4 Covering of ellipsoids

### 4.1 $s$ -numbers

Now we restrict our attention to the Hilbert space case, that is, we consider sets  $K$  of the form  $A(B_{\mathcal{H}})$  where  $A$  is an infinite-dimensional bounded compact operator from a Hilbert space  $\mathcal{H}$  to a Hilbert space  $\mathcal{H}_1$ . Such sets are called *ellipsoids*.

**Note.** We continue using the Banach space theory notation and terminology. In particular, unless explicitly stated otherwise, by  $A^*$  we mean the Banach-space-theoretical conjugate operator. It does not seem that anything will be gained if we introduce Hilbert-space duality, but it can cause some confusion when we apply Banach space case results for Hilbert spaces.

**Remark 4.1.** *Many of the results below are true for  $A(B_{\mathcal{H}})$  with non-compact  $A$  and usually the corresponding proofs are much simpler. We restrict our attention to the compact case.*

**Definition 4.2.** (See [8, Chapter II, §2]) The eigenvalues of the operator  $(E^*E)^{1/2}$  (where  $E^*$  is the conjugate in the Hilbert space sense) are called the  $s$ -numbers of the operator  $E$ . Notation:  $\{s_n(E)\}_{n=1}^{\infty}$ .

With this notation we have the following equalities for  $n$ -widths:

$$d_n(A(B_{\mathcal{H}})) = s_{n+1}(A)$$

(see [8, Theorem 2.2, p. 31]).

For ellipsoids we have a converse to the Lemma 3.2.

**Lemma 4.3.** *If  $K_0, K$  are ellipsoids in Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ , respectively, and  $d_n(K) \leq C d_n(K_0)$  for some  $C > 0$  and all  $n \in \mathbb{N} \cup \{0\}$ , then there is an operator  $D \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  such that  $DK_0 \supset K$  and  $\|D\| \leq C$ .*

*Proof.* The result follows from the so-called *Schmidt expansion* of a compact operator (see [8, p. 28]), which implies that

$$K = A(B_{\mathcal{H}}) = \left\{ \sum_{n=1}^{\infty} \alpha_n s_n(A) h_n : \{\alpha_n\}_{n=1}^{\infty} \in \ell_2, \{h_n\}_{n=1}^{\infty} \text{ is an orthonormal sequence} \right\}$$

and

$$K_0 = B(B_{\mathcal{H}}) = \left\{ \sum_{n=1}^{\infty} \alpha_n s_n(B) g_n : \{\alpha_n\}_{n=1}^{\infty} \in \ell_2, \{g_n\}_{n=1}^{\infty} \text{ is an orthonormal sequence} \right\}.$$

It is easy to see that there is a bounded linear operator  $D$  which maps  $g_n$  onto  $Ch_n$ , and that this operator satisfies the conditions  $D(K_0) \supset K$ ,  $\|D\| \leq C$ .  $\square$

**Remark 4.4.** *The proof of Lemma 4.3 shows that the desired operator  $D$  can be constructed as an operator whose restriction to  $V_{K_0}$  is a multiple of a suitable chosen bijective isometry between  $V_{K_0}$  and  $V_K$ , extended to  $\mathcal{H}_1$  in an arbitrary way.*

Known results on  $s$ -numbers imply the following lemma.

**Lemma 4.5.** *Let  $K$  be an ellipsoid in a Hilbert space  $\mathcal{H}$  such that  $\{d_n(K)\}_{n=0}^{\infty}$  is not lacunary. Let  $K_0$  be the intersection of  $K$  with a closed linear subspace of finite codimension. Then there exists  $\delta > 0$  such that  $d_n(K_0) \geq \delta d_n(K)$  and a bounded linear operator  $Q : \overline{\text{lin}(K_0)} \rightarrow \overline{\text{lin}(K)}$  satisfying  $Q(K_0) \supset K$ .*

*Proof.* Let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be a compact operator satisfying  $K = A(B_{\mathcal{H}})$ . The sequence  $\{d_n(K_0)\}_{n=0}^{\infty}$  is the sequence of  $s$ -numbers of a restriction of  $A$  to a subspace of finite codimension. This sequence is, in turn, the sequence of  $s$ -numbers of an operator of the form  $A + G$ , where  $G$  is an operator of finite rank.

It is known [8, Corollary 2.1, p. 29] that  $s_n(A + G) \geq s_{n+r}(A)$ , where  $r$  is the rank of  $G$ . Combining this inequality with the assumption that the sequence  $\{s_n(A)\}_{n=1}^{\infty}$  is not lacunary, we get the desired inequality.

The last statement of the lemma follows from Lemma 4.3.  $\square$

## 4.2 WOT

**Theorem 4.6.** *If  $\mathcal{H}$  is a Hilbert space,  $K \subset \mathcal{H}$  is an ellipsoid and the sequence  $\{d_n(K)\}$  is not lacunary, then  $WG(K) = \mathcal{L}(\mathcal{H})$ .*

The proof of Theorem 2.1 shows that to prove Theorem 4.6 it suffices to prove the following lemma (this can also be seen from the definition of WOT).

**Lemma 4.7.** *Let  $K$  be an ellipsoid in a Hilbert space  $\mathcal{H}$ . Suppose that the sequence  $\{d_n(K)\}$  is non-lacunary. Then for each finite-dimensional subspace  $\mathcal{Y} \subset \mathcal{H}$  and each linear mapping  $N : \mathcal{Y} \rightarrow \mathcal{H}$ , there is an operator  $D$  satisfying conditions:  $Dy = Ny$  for all  $y \in \mathcal{Y}$ , and  $DK \supset K$ .*

*Proof.* Let  $\mathcal{Z} = \mathcal{Y}^{\perp}$  and  $K_0 = K \cap \mathcal{Z}$ . By Lemma 4.5 there is an operator  $E$  from  $\mathcal{Z}$  to  $\mathcal{H}$  with  $EK_0 \supset K$ . Extend it to an operator  $D : \mathcal{H} \rightarrow \mathcal{H}$  setting  $Dy = Ny$  on  $\mathcal{Y}$ .  $\square$

**Remark 4.8.** *One can see from the proof of Lemma 4.7 that under the stated conditions the closure of  $G(K)$  in the strong operator topology coincides with  $\mathcal{L}(\mathcal{H})$ .*

**Corollary 4.9.** *Let  $K$  be an ellipsoid in a Hilbert space  $\mathcal{H}$ . Then:*

- (1)  $WG(K) = \mathcal{A}_K$  if the sequence  $\{d_n(K)\}_{n=0}^{\infty}$  is lacunary.
- (2)  $WG(K) = \mathcal{L}(\mathcal{H})$  if the sequence  $\{d_n(K)\}_{n=0}^{\infty}$  is not lacunary.

## 4.3 Ultra-weak topology

It turns out that Theorem 4.6 remains true if we replace closure in the weak operator topology, by a closure in a stronger topology, usually called *ultra-weak topology*. This topology on  $\mathcal{L}(\mathcal{H})$  is defined as the weak\* topology corresponding to the duality  $\mathcal{L}(\mathcal{H}) = (C_1(\mathcal{H}))^*$ , where  $C_1(\mathcal{H})$  is the space of nuclear operators. (Necessary background can be found in [18, Chapter II], unfortunately the terminology and notation there is different, the ultra-weak topology is called  *$\sigma$ -weak topology*, see [18, p. 67]). Ultra-weak and strong operator topologies are incomparable, for this reason our next result does not follow from Remark 4.8.

**Theorem 4.10.** *If  $K$  is an ellipsoid in a Hilbert space  $\mathcal{H}$  and the sequence  $\{d_n(K)\}_{n=1}^{\infty}$  is not lacunary, then the ultra-weak closure of  $G(K)$  coincides with  $\mathcal{L}(\mathcal{H})$ .*

*Proof.* Let  $\{T_i\}_{i=1}^m$  be a finite collection of operators in  $C_1(\mathcal{H})$  and  $R \in \mathcal{L}(\mathcal{H})$ . It suffices to show that there is  $D \in \mathcal{L}(\mathcal{H})$  satisfying

$$\mathrm{tr}(DT_i) = \mathrm{tr}(RT_i) \text{ for } i = 1, \dots, m \text{ and } DK \supset K. \quad (5)$$

It is clear that we may assume that the operators  $T_i$  are linearly independent.



**Lemma 4.11.** *If  $\{T_i\}_{i=1}^m$  are linearly independent, then there exists a finite rank projection  $P \in \mathcal{L}(\mathcal{H})$  such that the mapping*

$$\omega : \mathcal{L}(\mathcal{H}) \rightarrow \mathbb{R}^m$$

*given by*

$$\omega(U) = \{\text{tr}(UPT_i)\}_{i=1}^m$$

*is surjective.*

*Proof.* We have to prove that there is a finite rank projection  $P$  such that the operators  $PT_i$  are linearly independent (in this case the mapping  $\omega$  will be surjective). Using induction we may suppose that  $P_0T_1, \dots, P_0T_{m-1}$  are linearly independent for some  $P_0$ . Consider the set  $M_0$  of those finite rank projections  $P$  which commute with  $P_0$  and satisfy  $\text{im } P \supset \text{im } P_0$ . We claim that there exists  $P \in M_0$  such that  $PT_1, \dots, PT_m$  are linearly independent.

Assume contrary, then for each  $P \in M_0$ , one can find  $\lambda_1(P), \dots, \lambda_{m-1}(P) \in \mathbb{C}$  satisfying  $PT_m = \sum_{k=1}^{m-1} \lambda_k(P)PT_k$  (using the definition of  $M_0$  it is easy to get a contradiction if  $PT_1, \dots, PT_{m-1}$  are linearly dependent). Our next step is to show that the numbers  $\{\lambda_k(P)\}_{k=1}^{m-1}$  do not depend on  $P$ . In fact, for any  $P_1, P_2 \in M_0$  we have  $\sum_{k < m} (\lambda_k(P_1) - \lambda_k(P_2))P_0T_k = 0$ . So let  $\{\lambda_k\}_{k=1}^{m-1}$  be such that  $\lambda_k(P) = \lambda_k$  for all  $P \in M_0$ . Then the operator  $T = T_m - \sum_{k < m} \lambda_k T_k$  has the property that  $PT = 0$  for all  $P \in M_0$ . It is easy to see that this implies  $T = 0$ . We get a contradiction with the linear independence of  $\{T_k\}_{k=1}^m$ .  $\square$

We complete the proof of the theorem by showing the existence of  $D$  satisfying (5).

1. We define  $D$  on  $\ker P$  as in Theorem 4.6. This definition implies that the condition  $DK \supset K$  is satisfied.
2. To show that the condition  $\text{tr}(DT_i) = \text{tr}(RT_i)$ ,  $i = 1, \dots, m$ , is satisfied it suffices to show the existence of  $U \in \mathcal{L}(\mathcal{H})$  satisfying

$$\text{tr}((UP + D(I - P))T_i) = \text{tr}(RT_i) \quad \forall i = 1, \dots, m. \quad (6)$$

Since the condition (6) can be rewritten as  $\{\text{tr}(UPT_i)\}_{i=1}^m = \{\text{tr}((R - D(I - P))T_i)\}_{i=1}^m$ , where the right-hand side does not depend on  $U$ , and the vectors  $\{\text{tr}(UPT_i)\}_{i=1}^m$ ,  $U \in \mathcal{L}(\mathcal{H})$  cover (by Lemma 4.11) the whole space  $\mathbb{R}^m$ , the existence of  $U$  satisfying (6) follows.  $\square$

**Remark 4.12.** *It would be interesting to prove an analogue of Theorem 2.1 for the ultra-weak topology.*

## 5 Two ellipsoids

Let  $\mathcal{H}_1, \mathcal{H}_2$  be two infinite dimensional separable Hilbert spaces. We consider two ellipsoids,  $K_1 \subset \mathcal{H}_1$ ,  $K_2 \subset \mathcal{H}_2$  and introduce the set

$$G(K_1, K_2) := \{T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : TK_1 \supset K_2\}. \quad (7)$$

We are interested in the description of the WOT-closure of  $G(K_1, K_2)$  which we denote by  $WG(K_1, K_2)$ . As in the case of one ellipsoid, the description depends on the behavior of sequences of Kolmogorov  $n$ -widths.

We start with some simple but useful observations. It is easy to see that

$$G(K_2)G(K_1, K_2)G(K_1) \subset G(K_1, K_2).$$

Using this inclusion and elementary properties of WOT we get

$$WG(K_2)WG(K_1, K_2)WG(K_1) \subset WG(K_1, K_2). \quad (8)$$

Lemmas 3.2 and 4.3 imply that the set  $G(K_1, K_2)$  is non-empty if and only if

$$d_n(K_2) = O(d_n(K_1)). \quad (9)$$

From now on till the end of this section we assume that (9) is satisfied.

**Observation 5.1.** *By Remark 4.4, condition (9) implies that there is an onto isometry  $M : V_{K_1} \rightarrow V_{K_2}$  and a number  $\alpha \in \mathbb{R}^+$  such that  $\alpha M(K_1) \supset K_2$ . Consider decompositions  $\mathcal{H}_1 = V_{K_1} \oplus \mathcal{R}_1$  and  $\mathcal{H}_2 = V_{K_2} \oplus \mathcal{R}_2$ . Let  $A_1 \in \mathcal{L}(V_{K_1})$ ,  $B_1 \in \mathcal{L}(\mathcal{R}_1, \mathcal{H}_1)$ ,  $A_2 \in \mathcal{L}(V_{K_2})$ ,  $B_2 \in \mathcal{L}(\mathcal{R}_2, \mathcal{H}_2)$ , and  $C : \mathcal{R}_1 \rightarrow \mathcal{H}_2$ . Combining Theorem 2.1 with (8) we get that the composition  $(A_2 \oplus B_2)(\alpha M \oplus C)(A_1 \oplus B_1)$  is in  $WG(K_1, K_2)$ , where  $A_i \oplus B_i : V_{K_i} \oplus \mathcal{R}_i \rightarrow \mathcal{H}_i$ ,  $i = 1, 2$ .*

To state our results on the description of  $WG(K_1, K_2)$  we need the following definitions.

**Definition 5.2.** The  $k^{\text{th}}$  left shift of a sequence  $\{a_n\}_{n=0}^\infty$  ( $k \geq 0$ ) is the sequence  $\{a_{n+k}\}_{n=0}^\infty$ .

**Definition 5.3.** Let  $\{a_n\}_{n=0}^\infty$  and  $\{b_n\}_{n=0}^\infty$  be sequences of non-negative numbers. We say that  $\{a_n\}_{n=0}^\infty$  majorizes  $\{b_n\}_{n=0}^\infty$  if there is  $0 < C < \infty$  such that  $b_n \leq C a_n$  for all  $n = 0, 1, 2, \dots$ .

The following theorem is the main result of this section:

**Theorem 5.4.** *Let  $K_1$  and  $K_2$  be infinite dimensional ellipsoids in Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . Assume that (9) holds. Then*

- (A) *If all left shifts of the sequence  $\{d_n(K_1)\}_{n=0}^\infty$  majorize the sequence  $\{d_n(K_2)\}_{n=0}^\infty$ , then  $WG(K_1, K_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .*
- (B) *If the  $k^{\text{th}}$  left shift of  $\{d_n(K_1)\}$  majorizes the sequence  $\{d_n(K_2)\}$ , but the  $(k+1)^{\text{th}}$  left shift does not (such cases are clearly possible), then  $WG(K_1, K_2)$  is the set of those operators  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  for which the image of the space  $T(V_{K_1})$  in the quotient space  $\mathcal{H}_2/V_{K_2}$  is at most  $k$ -dimensional.*

*Proof.* (A) Observe that to show  $WG(K_1, K_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  it suffices to find, for an arbitrary finite-dimensional subspace  $\mathcal{Y} \in \mathcal{H}_1$  and an arbitrary operator  $N : \mathcal{Y} \rightarrow \mathcal{H}_2$ , an operator  $D \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  satisfying the conditions:  $D|_{\mathcal{Y}} = N|_{\mathcal{Y}}$  and  $D(K_1) \supset K_2$ . (This condition implies that  $G(K_1, K_2)$  is dense in  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  even in the strong operator topology.)

We find such an operator  $D$  in the following way: let  $\mathcal{Y}^\perp$  be an orthogonal complement of  $\mathcal{Y}$ . The argument of Lemma 4.5 shows that the sequence  $\{d_n(K_1 \cap \mathcal{Y}^\perp)\}$  majorizes some left shift of the sequence  $\{d_n(K_1)\}$  and thus, by our assumption, majorizes the sequence  $\{d_n(K_2)\}$ . By Lemma 4.3 there is a continuous linear operator  $Y : \mathcal{Y}^\perp \rightarrow \mathcal{H}_2$  such that  $Y(K_1 \cap \mathcal{Y}^\perp) \supset K_2$ . We let  $D|_{\mathcal{Y}^\perp} = Y$  and  $D|_{\mathcal{Y}} = N$ . It is clear that  $D$  has the desired properties.

(B) Suppose that the  $k^{\text{th}}$  left shift of  $\{d_n(K_1)\}$  majorizes  $\{d_n(K_2)\}$ . Let  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  be such that the dimension of the image of the space  $T(V_{K_1})$  in the quotient space  $\mathcal{H}_2/V_{K_2}$  is  $\leq k$ . We show that  $T \in WG(K_1, K_2)$ .

Let  $F$  be a finite subset of  $\mathcal{H}_2^*$  and  $\mathcal{Y}$  be a finite subset of  $\mathcal{H}_1$ . It suffices to show that there exists  $D \in G(K_1, K_2)$  satisfying  $f(Dy) = f(Ty)$  for each  $y \in \mathcal{Y}$  and each  $f \in F$ .

With this in mind, we may assume that  $F$  and  $\mathcal{Y}$  are finite dimensional subspaces. Also, we may assume that  $\mathcal{Y}$  is a subspace of  $V_{K_1}$ , because we may let the restriction of  $D$  to the orthogonal complement of  $V_{K_1}$  be the same as the restriction of  $T$ .

We decompose  $F$  as  $F_O \oplus F_V$ , where  $F_O = F \cap V_{K_2}^\perp$ . It is easy to check that the assumption on  $T$  implies that  $(T^*F_O)^\perp \cap V_{K_1}$  is of codimension at most  $k$  (if it is of codimension  $\geq k+1$ , then we can find  $k+1$  vectors  $x_i \in V_{K_1}$  and  $k+1$  functionals  $x_j^*$  in  $F_O$  such that  $T^*x_j^*(x_i) = \delta_{i,j}$ , but then  $x_j^*(Tx_i) = \delta_{ij}$  shows that  $\{Tx_i\}$  is a family of  $k+1$  vectors whose images in  $\mathcal{H}_2/V_{K_2}$  are linearly independent, contrary to our assumption).

Now we decompose  $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ , where  $\mathcal{Y}_1 = \mathcal{Y} \cap (T^*F_O)^\perp$ . We let  $D|_{\mathcal{Y}_2} = T|_{\mathcal{Y}_2}$ . Our next step is to find a suitable definition of the restriction of  $D$  to  $(T^*F_O)^\perp \cap V_{K_1}$ . To this end we need the following modification of Lemma 4.5, which can be proved using the same argument and Remark 4.4.

**Lemma 5.5.** *Let  $K_1$  and  $K_2$  be ellipsoids in Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Suppose that the  $k^{\text{th}}$  left shift of  $\{d_n(K_1)\}_{n=0}^\infty$  majorizes  $\{d_n(K_2)\}_{n=0}^\infty$  and that  $K_0$  is the intersection of  $K_1$  with a subspace of  $\mathcal{H}_1$  of codimension  $k$ . Then there exists an operator  $B : V_{K_0} \rightarrow V_{K_2}$  such that  $B(K_0) \supset K_2$  and  $B$  is a multiple of a bijective linear isometry of  $V_{K_0}$  and  $V_{K_2}$ .*

Applying Lemma 5.5 we find an operator  $B : ((T^*F_O)^\perp \cap V_{K_1}) \rightarrow V_{K_2}$  which satisfies  $B((T^*F_O)^\perp \cap V_{K_1}) \supset K_2$  and is a multiple of a bijective isometry. Now we modify  $B$  using Lemma 2.3, which we apply for  $\mathcal{X} = V_{K_2}$ ,  $K = K_2$ ,  $\mathcal{Y} = B\mathcal{Y}_1$ ,  $N = TB^{-1}|_{B\mathcal{Y}_1}$ , and  $F$  (which is denoted in the same way in this proof). We denote the operator obtained as a result of the application of Lemma 2.3 by  $H$ .

We let  $D|_{(T^*F_O)^\perp \cap V_{K_1}} = HB$ . This formula defines  $D$  on  $\mathcal{Y}_1$ , and this definition is such that  $D|_{\mathcal{Y}_1} = T|_{\mathcal{Y}_1}$ . We extend  $D$  to the rest of the space  $\mathcal{H}_1$  arbitrarily.

It is clear that  $D$  satisfies all the assumptions. Thus  $T \in WG(K_1, K_2)$ .

Now we suppose that the  $(k+1)^{\text{th}}$  left shift of  $\{d_n(K_1)\}$  does not majorize  $\{d_n(K_2)\}$  and show that if  $T$  is an operator for which  $T(V_{K_1})$  contains  $k+1$  vectors whose images in the quotient space  $\mathcal{H}_2/V_{K_2}$  are linearly independent, then  $T \notin WG(K_1, K_2)$ .

Using the standard argument we find  $v_1, \dots, v_{k+1} \in V_{K_1}$ , functionals  $f_1, \dots, f_{k+1} \in \mathcal{H}_2^*$ , and  $\varepsilon > 0$  such that any  $D \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  satisfying  $|f_j(Dv_i - Tv_i)| < \varepsilon$ ,  $i, j = 1, \dots, k+1$ , satisfies the condition:  $D(V_{K_1})$  contains  $k+1$  vectors whose images in  $\mathcal{H}_2/V_{K_2}$  are linearly independent. It remains to show that such operators  $D$  cannot satisfy  $DK_1 \supset K_2$ .

In fact the condition about  $k+1$  linearly independent vectors implies that  $D^{-1}(V_{K_2}) \cap V_{K_1}$  is a subspace of  $V_{K_1}$  of codimension at least  $k+1$ .

Therefore  $K_2$  is covered by a section  $K_0$  of  $K_1$  of codimension  $k+1$ . On the other hand, by Lemma 3.3, the sequence of  $n$ -widths of  $K_0$  is majorized by the  $(k+1)^{\text{th}}$  left shift of  $\{d_n(K_1)\}_{n=1}^\infty$ . By Lemma 3.2, we get a contradiction with our assumption.  $\square$

**Corollary 5.6.** *If  $\{d_n(K_1)\}_{n=0}^\infty$  is non-lacunary and the condition (9) is satisfied, then  $WG(K_1, K_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .*

In fact, if  $\{d_n(K_1)\}_{n=0}^\infty$  is non-lacunary, it is majorized by each of its left shifts, and hence the assumption of Theorem 5.4(A) is satisfied.

**Remark 5.7.** *In the case where  $\{d_n(K_1)\}_{n=0}^\infty$  is lacunary both the situation described in Theorem 5.4(A) and the situation described in Theorem 5.4(B) can occur.*

Similarly to the case of one compact we introduce

$$\mathcal{A}_{K_1, K_2} := \{T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) : TV_{K_1} \subset V_{K_2}\}.$$

The following is a special case of Theorem 5.4(B) corresponding to the case  $k = 0$ :

**Corollary 5.8.** *Let  $K_1, K_2$  be ellipsoids with*

$$\liminf \frac{d_{n+1}(K_1)}{d_n(K_2)} = 0. \quad (10)$$

*Then  $WG(K_1, K_2) = \mathcal{A}_{K_1, K_2}$ .*

**Remark 5.9.** *Note that the combination of the assumption (9) and the condition (10) imply that the sequences  $\{d_n(K_1)\}_{n=1}^\infty$  and  $\{d_n(K_2)\}_{n=1}^\infty$  are both lacunary. Indeed,  $d_k(K_2) \leq Cd_k(K_1)$  implies*

$$\frac{d_{n+1}(K_1)}{d_n(K_2)} \geq \frac{d_{n+1}(K_1)}{Cd_n(K_1)} \text{ and } \frac{d_{n+1}(K_1)}{d_n(K_2)} \geq \frac{d_{n+1}(K_2)}{Cd_n(K_2)}.$$

*Therefore (10) implies that  $\{d_n(K_1)\}_{n=1}^\infty$  and  $\{d_n(K_2)\}_{n=1}^\infty$  are lacunary.*

Analysis of all possible cases in Theorem 5.4 implies also the following:

**Corollary 5.10.** *If  $K_1$  and  $K_2$  are ellipsoids for which  $V_{K_i} = \mathcal{H}_i$  for  $i = 1, 2$ , and (9) is satisfied, then  $WG(K_1, K_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .*

## 6 Covering with compact operators

Here we discuss the problem of covering an ellipsoid  $K_2$  by the image of an ellipsoid  $K_1$  via a compact operator. Let  $CG(K_1, K_2)$  be the set of all compact operators  $T$  satisfying the condition  $TK_1 \supset K_2$ .

Let us begin with an analogue of Lemma 3.2.

Note that the widths  $d_n(K)$  of a compact subset  $K$  in a Banach space  $\mathcal{X}$  can change if we consider  $K$  as a subset of a subspace  $\mathcal{Y} \subset \mathcal{X}$  that contains  $K$ . Let us denote by  $\tilde{d}_n(K)$  the  $n$ -width of  $K$  considered as a subset of  $V_K$  (recall that  $V_K = \overline{\text{lin } K}$ , so we choose the minimal subspace and obtain maximal widths).

**Lemma 6.1.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be Banach spaces,  $K$  be a compact set in  $\mathcal{X}$  and  $T : \mathcal{X} \rightarrow \mathcal{Y}$  be a compact operator. Then  $\tilde{d}_n(TK)/\tilde{d}_n(K) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* We may assume that  $\mathcal{X} = V_K$ . By the definition of  $\tilde{d}_n$ , for each  $n \in \mathbb{N} \cup \{0\}$  and  $0 < \varepsilon < 1$ , there exists an  $n$ -dimensional subspace  $\mathcal{X}_n \subset \mathcal{X}$  such that

$$K \subset \mathcal{X}_n + d_n(K)(1 + \varepsilon)B_{\mathcal{X}}. \quad (11)$$

Therefore

$$TK \subset T\mathcal{X}_n + d_n(K)(1 + \varepsilon)TB_{\mathcal{X}}. \quad (12)$$

Now we show that for each  $\delta > 0$  there is  $N \in \mathbb{N}$  such that

$$TB_{\mathcal{X}} \subset T\mathcal{X}_n + \delta B_{\mathcal{Y}} \text{ for } n \geq N. \quad (13)$$

In fact, since  $\overline{TB_{\mathcal{X}}}$  is compact, it has a finite  $\delta/3$ -net  $\{y_i\}_{i=1}^t \subset TB_{\mathcal{X}}$ . Since  $TB_{\mathcal{X}} \subset TV_K$ , the vectors  $y_i$  can be arbitrarily well approximated by linear combinations of vectors

from  $TK$ . Let  $M$  be the maximum absolute sum of coefficients of a selection of such  $\delta/3$ -approximating linear combinations. Let  $N$  be such that for  $n \geq N$  we have  $d_n(K) \leq \frac{\delta}{6M\|T\|}$ , and let us show that (13) holds.

We need to show that for all  $y \in TB_{\mathcal{X}}$  we have  $\text{dist}(y, T\mathcal{X}_n) \leq \delta$ .

Let  $j \in \{1, \dots, t\}$  be such that  $\|y - y_j\| < \delta/3$ , and let  $\sum_{i=1}^s \alpha_i T x_i$  be such that  $x_i \in K$ ,  $\sum_{i=1}^s |\alpha_i| \leq M$ , and  $\|y_j - \sum_{i=1}^s \alpha_i T x_i\| < \delta/3$ . By (11), we have  $\text{dist}(x_i, \mathcal{X}_n) \leq d_n(K)(1 + \varepsilon)$ . Therefore  $\text{dist}(\sum_{i=1}^s \alpha_i T x_i, T\mathcal{X}_n) \leq \sum_{i=1}^s |\alpha_i| \|T\| d_n(K)(1 + \varepsilon) \leq M\|T\| \cdot \frac{\delta}{6M\|T\|} \cdot (1 + \varepsilon) < \frac{\delta}{3}$ . Thus  $\text{dist}(y, T\mathcal{X}_n) < \delta$ .

If we combine (12) and (13) we get  $d_n(TK) \leq (1 + \varepsilon)\delta d_n(K)$  for  $n \geq N$ . Since  $0 < \varepsilon < 1$  and  $\delta > 0$  can be chosen arbitrarily, the statement follows.  $\square$

**Remark 6.2.** Note that if  $\mathcal{X}$  is a Hilbert space, then  $\tilde{d}_n(K) = d_n(K)$ . Indeed, in this case we may assume that  $\mathcal{X}_n \subset V_K$ . Such subspace can be found as the orthogonal projection to  $V_K$  of any subspace  $\mathcal{X}_n$  satisfying (11). It should be mentioned that in the Hilbert space case a simpler proof of Lemma 6.1 is known, see [7, Lemma 1].

Now we find criteria of non-emptiness of  $CG(K_1, K_2)$  for ellipsoids  $K_1$  and  $K_2$ . The result can be considered as an analogue of Lemmas 3.2 and 4.3.

**Lemma 6.3.** Let  $K_1$  and  $K_2$  be ellipsoids in Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. There is a compact operator  $T$  satisfying  $TK_1 \supset K_2$  if and only if

$$d_n(K_2) = o(d_n(K_1)). \quad (14)$$

*Proof.* If there is a compact operator  $T$  with  $TK_1 \supset K_2$  then (14) follows from Lemma 6.1 and Remark 6.2. Conversely, if (14) holds, then the existence of a *compact* operator  $T$  follows from the argument of Lemma 4.3.  $\square$

If the condition (14) is satisfied we say: the sequence  $\{d_n(K_1)\}_{n=1}^{\infty}$  *strictly majorizes*  $\{d_n(K_2)\}_{n=1}^{\infty}$ .

Let us define by  $WCG(K_1, K_2)$  the WOT-closure of  $CG(K_1, K_2)$ . We have the following analogue of Theorem 5.4:

**Theorem 6.4. (A)** If all left shifts of the sequence  $\{d_n(K_1)\}$  strictly majorize the sequence  $\{d_n(K_2)\}$ , then  $WCG(K_1, K_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .

**(B)** If the  $k^{\text{th}}$  left shift of  $\{d_n(K_1)\}$  strictly majorizes the sequence  $\{d_n(K_2)\}$ , but the  $(k+1)^{\text{th}}$  left shift does not (such cases are clearly possible), then  $WCG(K_1, K_2)$  is the set of those operators  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  for which the image of the space  $T(V_{K_1})$  in the quotient space  $\mathcal{H}_2/V_{K_2}$  is at most  $k$ -dimensional.

The proof is a straightforward modification of the proof of Theorem 5.4 and we omit it.

## 7 Operator ranges

In this section by a Hilbert space we mean a separable infinite dimensional Hilbert space. An operator range is the image of a Hilbert space  $\mathcal{H}_1$  under a bounded operator  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ . Operator ranges are actively studied, see [2], [4], [9], [13], and references therein. The purpose of this section is to use the results of the previous section to classify operator ranges. Our results complement the classification of operator ranges presented in [4, Section 2].

We restrict our attention to images of *compact operators of infinite rank*. The set  $A(B_{\mathcal{H}_1})$  will be called a *generating ellipsoid* of the operator range  $A\mathcal{H}_1$ . The same operator range is the image of infinitely many different operators, therefore a generating ellipsoid of an operator range is not uniquely determined. However, the Baire category theorem implies that if  $K_1$  and  $K_2$  are generating ellipsoids of the same operator range, then  $cK_1 \subset K_2 \subset CK_1$  for some  $0 < c \leq C < \infty$ .

We say that two sequences of positive numbers are *equivalent* if each of them majorizes the other. The observation above implies that the equivalence class of the sequence of  $n$ -widths  $\{d_n(K)\}_{n=0}^\infty$  of a generating ellipsoid of  $\mathcal{Y}$  is uniquely determined by an operator range  $\mathcal{Y}$ . We denote this equivalence class of sequences by  $d(\mathcal{Y})$ .

It is clear that a sequence is lacunary if and only if all of sequences equivalent to it are lacunary. It is also clear that left shifts of equivalence classes of sequences are well-defined as well as the conditions like  $d(\mathcal{Y}_1)$  *majorizes*  $d(\mathcal{Y}_2)$ . Therefore the following notions are well-defined for operator ranges: (i)  $\mathcal{Y}$  is *lacunary*; (ii)  $\mathcal{Y}_1$  *majorizes*  $\mathcal{Y}_2$ . We say that an operator range  $\mathcal{Y} \subset \mathcal{H}$  is *dense* if  $\overline{\mathcal{Y}} = \mathcal{H}$ .

Results of Section 5 on covering of one ellipsoid by another have immediate corollaries for operator ranges. Let  $A_1 : \mathcal{H} \rightarrow \mathcal{H}_1$  and  $A_2 : \mathcal{H} \rightarrow \mathcal{H}_2$  be compact operators of infinite rank and  $\mathcal{Y}_i = A_i\mathcal{H}$ . Let  $\mathcal{R}(\mathcal{Y}_1, \mathcal{Y}_2)$  denote the set of all operators  $T$  satisfying

$$T\mathcal{Y}_1 \supset \mathcal{Y}_2. \quad (15)$$

We write  $\mathcal{R}(\mathcal{Y})$  instead of  $\mathcal{R}(\mathcal{Y}, \mathcal{Y})$ . The WOT-closure of  $\mathcal{R}(\mathcal{Y}_1, \mathcal{Y}_2)$  will be denoted by  $\mathcal{WR}(\mathcal{Y}_1, \mathcal{Y}_2)$ .

**Corollary 7.1.** *Suppose that  $\mathcal{Y}_1$  majorizes  $\mathcal{Y}_2$ . Then*

- (i) *If all left shifts of  $d(\mathcal{Y}_1)$  majorize  $d(\mathcal{Y}_2)$ , then  $\mathcal{WR}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .*
- (ii) *Let  $k$  be a non-negative integer. If the  $k^{\text{th}}$  left shift of  $d(\mathcal{Y}_1)$  majorizes  $d(\mathcal{Y}_2)$ , but the  $(k+1)^{\text{th}}$  left shift does not, then  $\mathcal{WR}(\mathcal{Y}_1, \mathcal{Y}_2)$  is the set of those operators  $T$  for which the image of  $T\mathcal{Y}_1$  in the quotient space  $\mathcal{H}_2/\overline{\mathcal{Y}_2}$  has dimension  $\leq k$ . In particular, if  $k = 0$ , we get: if the first left shift of  $d(\mathcal{Y}_1)$  does not majorize  $d(\mathcal{Y}_2)$ , then  $\mathcal{WR}(\mathcal{Y}_1, \mathcal{Y}_2)\overline{\mathcal{Y}_1} \subset \overline{\mathcal{Y}_2}$ .*
- (iii) *If  $\mathcal{Y}_1$  is non-lacunary, then  $\mathcal{WR}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .*
- (iv) *If  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  are dense, then  $\mathcal{WR}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ .*

*Proof.* To derive (i)-(iv) from Theorem 5.4 and its corollaries we need two observations:

- $\mathcal{R}(\mathcal{Y}_1, \mathcal{Y}_2)$  contains  $G(K_1, K_2)$  for any pair of generating ellipsoids.
- If  $T \in \mathcal{R}(\mathcal{Y}_1, \mathcal{Y}_2)$  then  $TK_1 \supset K_2$  for some pair of generating ellipsoids.

The first observation immediately implies (i), (iii), (iv), and “estimates from below” in (iv). The second observation shows that for “estimates from above” in (ii) we can use the same argument as in Section 5.  $\square$

One of the systematically studied objects in the theory of invariant subspaces, see [5, 14, 15, 16], is the algebra  $\mathcal{A}(\mathcal{Y})$  of all operators that preserve invariant a given operator range  $\mathcal{Y}$ . It is known, see [16, Theorem 1], that if  $\mathcal{Y}$  is dense, then the WOT-closure  $\mathcal{WA}(\mathcal{Y})$  of  $\mathcal{A}(\mathcal{Y})$  coincides with  $\mathcal{L}(\mathcal{H})$ . It follows easily that in general  $\mathcal{WA}(\mathcal{Y})$  consists of all operators that preserve the closure  $\overline{\mathcal{Y}}$  of  $\mathcal{Y}$ .

An operator algebra  $\mathcal{A}$  is called *full* if it contains the inverses of all invertible operators in  $\mathcal{A}$ . We call  $\mathcal{A}$  *weakly full* if for each invertible operator  $T \in \mathcal{A}$ , the operator  $T^{-1}$  belongs to the WOT-closure of  $\mathcal{A}$ . Our next result shows that for algebras of the form  $\mathcal{A}(\mathcal{Y})$  this property depends on  $d(\mathcal{Y})$ .

**Corollary 7.2.** (i) *If the closure  $\overline{\mathcal{Y}}$  of an operator range  $\mathcal{Y} \subset \mathcal{H}$  has finite codimension in  $\mathcal{H}$ , then the algebra  $\mathcal{A}(\mathcal{Y})$  is weakly full.*

(ii) *If  $\mathcal{Y}$  is not lacunary and  $\text{codim}(\overline{\mathcal{Y}}) = \infty$ , then  $\mathcal{A}(\mathcal{Y})$  is not weakly full.*

(iii) *If  $\mathcal{Y}$  is lacunary, then  $\mathcal{A}(\mathcal{Y})$  is weakly full.*

*Proof.* (i) If  $T$  preserves  $\mathcal{Y}$  then  $T\overline{\mathcal{Y}} \subset \overline{\mathcal{Y}}$ . If  $T$  is invertible, then it maps a complement of  $\overline{\mathcal{Y}}$  onto a complement of  $T(\overline{\mathcal{Y}})$ . If  $\overline{\mathcal{Y}}$  has finite codimension, this implies  $T\overline{\mathcal{Y}} = \overline{\mathcal{Y}}$ . Hence  $T^{-1}\overline{\mathcal{Y}} = \overline{\mathcal{Y}}$ , and  $T^{-1}$  is in the WOT-closure of  $\mathcal{A}(\mathcal{Y})$ .

(ii) Let  $K$  be a generating ellipsoid of  $\mathcal{Y}$ . Choose a nonzero vector  $y \in \mathcal{Y}$  and let  $K_0 = K \cap y^\perp$ . By Lemma 4.5, the sequences  $\{d_n(K)\}$  and  $\{d_n(K_0)\}$  are equivalent. Using Observation 5.1 we find an operator  $D : V_{K_0} \rightarrow V_K$  which satisfies  $D(K_0) \supset K$  and is a (nonzero) multiple of an isometry. Since  $\overline{\mathcal{Y}}$  has infinite codimension, we can extend  $D$  to an invertible operator  $D : \mathcal{H} \rightarrow \mathcal{H}$ . Observe that  $D(\overline{\mathcal{Y}} \cap y^\perp) = \overline{\mathcal{Y}}$ , therefore  $D(y) \notin \overline{\mathcal{Y}}$ , and thus  $D \notin \mathcal{WA}(\mathcal{Y})$ . On the other hand, the inclusion  $D(K_0) \supset K$  implies  $D^{-1} \in \mathcal{A}(\mathcal{Y})$ .

(iii) If  $T \in \mathcal{A}(\mathcal{Y})$  is invertible, then  $T^{-1} \in \mathcal{R}(\mathcal{Y})$ . Since  $\mathcal{Y}$  is lacunary, applying Corollary 7.1 we conclude that  $T^{-1}$  preserves  $\overline{\mathcal{Y}}$ . Therefore  $T^{-1} \in \mathcal{WA}(\mathcal{Y})$ .  $\square$

## 8 Bilinear operator equations

One of the popular topics in operator theory is the study of linear operator equations  $XA = B$  and  $AX = B$ . We consider here a “bilinear operator equation”

$$XAY = B, \tag{16}$$

where operators  $A, B$  are given. Its solution is a pair  $(X, Y)$  of operators. We denote the set of all such solutions by  $\mathcal{S}(A, B)$ . For simplicity we restrict our attention to the case when all operators act on a fixed separable Hilbert space  $\mathcal{H}$ . Such a pair  $(X, Y)$  can be found if we fix one of the operators (say  $X$ ) and solve the obtained linear equation (which has more than one solution in the degenerate cases only). So the study of the question “how many solutions does equation (16) have?” reduces to the study of the set of all first components, that is, the set of those  $X$  for which  $(X, Y)$  is a solution for some  $Y$ . Let us denote this set by  $U(A, B)$ .

**Corollary 8.1.** (i) *The equation is solvable if and only if*

$$s_n(B) = O(s_n(A)). \tag{17}$$

(ii) *Suppose that condition (17) holds. If operators  $A, B$  have dense ranges, or if the range of  $A$  is non-lacunary, then  $U(A, B)$  is WOT-dense in  $\mathcal{L}(\mathcal{H})$ .*

(iii) *If the range of operator  $B$  is not dense and the condition*

$$s_n(B) = O(s_{n+1}(A)) \tag{18}$$

*does not hold, then  $U(A, B)$  is not WOT-dense in  $\mathcal{L}(\mathcal{H})$ .*

*Proof.* Clearly  $X \in U(A, B)$  if and only if the equation (16) is solvable with respect to  $Y$ . This is equivalent to the inclusion  $XA\mathcal{H} \supset B\mathcal{H}$ . It remains to apply Corollary 7.1.  $\square$

If an operator  $A$  is not compact then the set is WOT-dense in  $\mathcal{L}(\mathcal{H})$ . Formally this is not a special case of Corollary 8.1(ii) because  $s$ -numbers are usually defined for compact operators only, but the proof in this case along the same lines is even simpler. In the rest of the section we prove that this result can be considerably strengthened: if  $A$  is not compact then  $\mathcal{S}(A, B)$  itself is dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$  with respect to the weak (and even strong) operator topology.

**Lemma 8.2.** *If we are given two linearly independent families  $(x_1, \dots, x_n)$ ,  $(y_1, \dots, y_m)$  of vectors in  $\mathcal{H}$ , two arbitrary families  $(x'_1, \dots, x'_n)$ ,  $(y'_1, \dots, y'_m)$  of vectors in  $\mathcal{H}$ , and a number  $\epsilon > 0$ , then there is an invertible operator  $V$  with the properties  $\|Vx_i - x'_i\| < \epsilon$ ,  $\|V^{-1}y_j - y'_j\| < \epsilon$ .*

*Proof.* One can choose systems  $z_1, \dots, z_n$  and  $w_1, \dots, w_m$  close to  $(x'_1, \dots, x'_n)$  and, respectively,  $(y'_1, \dots, y'_m)$  in such a way that both systems

$$(x_1, \dots, x_n, w_1, \dots, w_m) \text{ and } (y_1, \dots, y_m, z_1, \dots, z_n)$$

are linearly independent. Let us define an operator  $T$  between their linear spans by  $Tx_i = z_i, Tw_j = y_j$ . It is injective and therefore can be extended to an invertible operator on a finite dimensional space containing these systems. Clearly an invertible operator on a finite-dimensional subspace can be extended to an invertible operator on the whole space (take the direct sum with the identity operator).  $\square$

We denote the group of all invertible operators on  $\mathcal{H}$  by  $\mathcal{G}(\mathcal{H})$ . **Note.** In this section  $A^*$  denotes the Hilbert space conjugate of an operator  $A$ .

**Lemma 8.3.** *If an operator  $X$  has dense image and an operator  $Y$  has trivial kernel, then the set*

$$\Gamma_{X,Y} = \{(XV^{-1}, VY) : V \in \mathcal{G}(\mathcal{H})\}$$

*is dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$  with respect to the strong operator topology (SOT).*

*Proof.* Let a system  $(x_1, \dots, x_n)$ ,  $(y_1, \dots, y_m)$ ,  $(x'_1, \dots, x'_n)$ ,  $(y'_1, \dots, y'_m)$  and  $\epsilon > 0$  be given as above. The system  $\tilde{x}_i = Yx_i$  is linearly independent since  $\ker Y = 0$ . Since  $X\mathcal{H}$  is dense, there are  $z_j$  with  $\|Xz_j - y'_j\| < \epsilon/2$ . Take  $0 < \delta < \frac{\epsilon}{2\|X\|}$  and choose an invertible operator  $V$  as in Lemma 8.2 for the system  $(\tilde{x}_1, \dots, \tilde{x}_n)$ ,  $(y_1, \dots, y_m)$ ,  $(x'_1, \dots, x'_n)$ ,  $(z_1, \dots, z_m)$  and  $\delta$ . The obtained inequalities imply that  $\Gamma_{X,Y}$  is SOT-dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ .  $\square$

Any solution  $(X, Y)$  of the equation  $XY = B$  will be called a *factorization* of an operator  $B$ .

**Proposition 8.4.** *For each operator  $B$  in an infinite-dimensional Hilbert space  $\mathcal{H}$ , the set  $\mathcal{P}(B)$  of all its factorizations is SOT-dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ .*

*Proof.* Let  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are of the same dimension as  $\mathcal{H}$ . Let  $U_1$  and  $U_2$  be isometries with the ranges  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Then  $U_1^*$  and  $U_2^*$  isometrically map  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively, onto  $\mathcal{H}$ , also  $U_1^*\mathcal{H}_2 = \{0\}$  and  $U_2^*\mathcal{H}_1 = \{0\}$ . We set  $Y = U_1$  and  $X = BU_1^* + U_2^*$ .

Since  $XY = BU_1^*U_1 + U_2^*U_1 = B$ , we have  $(X, Y) \in \mathcal{P}(B)$ , and therefore  $(XV^{-1}, VY) \in \mathcal{P}(B)$  for each  $V \in \mathcal{G}(\mathcal{H})$ . It follows easily from the definition of operators  $X, Y$  that  $X\mathcal{H} = \mathcal{H}$  and  $\ker(Y) = 0$ . Applying Lemma 8.3 we conclude that  $\mathcal{P}(B)$  is SOT-dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ .  $\square$

Let us write  $A \succ B$  if the set  $\mathcal{S}(A, B)$  of all solutions of (16) is SOT-dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ . For brevity, we will denote by  $\overline{\mathcal{E}}^s$  the closure of a subset  $\mathcal{E}$  of  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$  with respect to the product of SOT-topologies.



**Lemma 8.5.** *If  $A \succ B$  and  $B \succ C$ , then  $A \succ C$ .*

*Proof.* For each  $(X, Y) \in \mathcal{S}(A, B)$  and each  $(X_1, Y_1) \in \mathcal{S}(B, C)$ , one has  $(XX_1, Y_1Y) \in \mathcal{S}(A, C)$ . Taking  $(X_1, Y_1) \rightarrow (I, I)$  we get that  $(X, Y) \in \overline{\mathcal{S}(A, C)}^s$ . Hence  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \subset \overline{\mathcal{S}(A, C)}^s$  and  $A \succ C$ .  $\square$

We proved in Proposition 8.4 that  $I \succ C$  for all  $C$ . So our aim is to show that  $A \succ I$  for each non-compact  $A$ .

**Lemma 8.6.** *If  $P$  is a projection of infinite rank, then  $P \succ I$ .*

*Proof.* Let  $U$  be an isometry with  $UU^* = P$ . Then  $(U^*, U) \in \mathcal{S}(P, I)$ . Hence

$$(VU^*, UV^{-1}) \in \mathcal{S}(P, I) \text{ for each } V \in \mathcal{G}(\mathcal{H}).$$

It follows that  $\overline{\mathcal{S}(P, I)}^s$  contains all pairs  $(M, N)$  with  $N\mathcal{H} \subset P\mathcal{H}$ ,  $M(I - P) = 0$ .

Hence for each  $(X, Y) \in \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$ , the pair  $(XP, PY)$  belongs to  $\overline{\mathcal{S}(P, I)}^s$ . Choose a net  $(X_\lambda, Y_\lambda)$  in  $\mathcal{S}(P, I)$  with  $(X_\lambda, Y_\lambda) \rightarrow (XP, PY)$  in SOT, then  $(X_\lambda + X(I - P), Y_\lambda + (I - P)Y) \in \mathcal{S}(P, I)$  (indeed  $(X_\lambda + X(I - P))P(Y_\lambda + (I - P)Y) = X_\lambda PY_\lambda = 1$ ). Since  $(X_\lambda + X(I - P), Y_\lambda + (I - P)Y) \rightarrow (X, Y)$  we get that  $(X, Y) \in \overline{\mathcal{S}(P, I)}^s$ .  $\square$

The proof of the next lemma is immediate.

**Lemma 8.7.** (i) *If  $(X, Y) \in \mathcal{S}(F_1AF_2, I)$ , then  $(XF_1, F_2Y) \in \mathcal{S}(A, I)$ .*

*In particular*

(ii) *If  $F_1AF_2 \succ I$ ,  $\ker(F_1) = 0$  and  $\overline{F_2\mathcal{H}} = \mathcal{H}$  then  $A \succ I$ .*

**Lemma 8.8.** *Let  $A = 0 \oplus A_1$ , where  $A_1$  acts on infinite-dimensional space and is invertible. Then  $A \succ I$ .*

*Proof.* Let  $F = I \oplus A_1^{-1}$  then  $F$  is invertible and  $FA$  is a projection of infinite rank. Hence  $FA \succ 1$ , by Lemma 8.6. Using Lemma 8.7 (ii), we get that  $A \succ 1$ .  $\square$

**Lemma 8.9.** *If  $A \geq 0$  and  $A$  is not compact, then  $A \succ I$ .*

*Proof.* For each  $\varepsilon > 0$ , let  $P_\varepsilon = I - Q$ , where  $Q$  is the spectral projection of  $A$  corresponding to the interval  $(0, \varepsilon)$ . Then  $P_\varepsilon A$  is of the form  $0 \oplus B$ , where  $B$  is invertible and, for sufficiently small  $\varepsilon$ , non-compact. Hence  $P_\varepsilon A \succ I$ . By Lemma 8.7,  $\mathcal{L}(\mathcal{H})P_\varepsilon \times \mathcal{L}(\mathcal{H}) \subset \overline{\mathcal{S}(A, I)}^s$ . Since  $P_\varepsilon \rightarrow I$  when  $\varepsilon \rightarrow 0$ , we get that  $A \succ I$ .  $\square$

**Theorem 8.10.** *If  $A$  is non-compact, then the set of all solutions of the equation (16) is SOT-dense in  $\mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H})$  for each  $B$ .*

*Proof.* It suffices to show that  $A \succ I$ . Suppose firstly that the operator  $U$  in the polar decomposition  $A = UT$  of  $A$  is an isometry. The operator  $AU^* = UTU^*$  is non-negative and non-compact. Hence  $AU^* \succ I$ . Since  $\overline{U^*\mathcal{H}} = \mathcal{H}$ ,  $A \succ I$ .

If  $U$  is a coisometry, then  $U^*A = T$  is a positive non-compact operator. So  $T \succ I$ , and since  $\ker(U^*) = 0$ , we get  $A \succ I$ .  $\square$

## 9 $A$ -expanding operators

In operator theory, especially in dealing with interpolation problems, one often needs to consider Hilbert (or Banach) spaces with two norms and study operators with special properties with respect to these norms. The main purpose of this section is to show that Kolmogorov  $n$ -widths can be used to describe WOT-closures of some sets of operators

given by conditions of this kind. Our interest to such conditions is inspired by the theory of linear fractional relations, see [11] and [12].

Let  $\mathcal{X}$  be a Banach space and  $A \in \mathcal{L}(\mathcal{X})$  be a compact operator with an infinite-dimensional range. It determines a semi-norm  $\|x\|_A = \|Ax\|$  on  $\mathcal{X}$ . We consider the set  $\mathcal{E}(A)$  of all operators  $T$  that increase this semi-norm:  $\|Tx\|_A \geq \|x\|_A$  for each  $x \in \mathcal{X}$ . In other words

$$\mathcal{E}(A) := \{T \in \mathcal{L}(\mathcal{X}) : \|ATx\| \geq \|Ax\| \ \forall x \in \mathcal{X}\}. \quad (19)$$

It turns out that the problem of description of  $\mathcal{E}(A)$  is a dual version of the problem considered in previous sections: the following dual characterization of  $\mathcal{E}(A)$  relates it with covering operators.

**Lemma 9.1.** *Let a Banach space  $\mathcal{X}$  be reflexive. An operator  $R \in \mathcal{L}(\mathcal{X})$  satisfies  $R \in \mathcal{E}(A)$  if and only if  $R^* \in G(K)$ , where  $K = A^*(B_{\mathcal{X}^*})$ .*

*Proof.* Assume that  $R^* \in G(K)$ , that is,  $R^*K \supset K$ . Then

$$\|ARx\| = \sup_{f \in B_{\mathcal{X}^*}} |f(ARx)| = \sup_{f \in B_{\mathcal{X}^*}} |(R^*A^*f)(x)| \geq \sup_{f \in B_{\mathcal{X}^*}} |(A^*f)(x)| = \|Ax\|, \quad (20)$$

for each  $x \in X$ . Thus  $R \in \mathcal{E}(A)$ .

Conversely, if  $R^* \notin G(K)$ , then there is  $f \in K \setminus R^*K$ . The set  $R^*K$  is weakly closed. By the Hahn–Banach theorem and reflexivity of  $\mathcal{X}$  there is  $x \in \mathcal{X}$  with  $|f(x)| > \sup_{g \in R^*K} |g(x)| = \|ARx\|$ . Since  $|f(x)| \leq \|Ax\|$  we obtain that  $R \notin \mathcal{E}(A)$ .  $\square$

We denote the WOT-closure of  $\mathcal{E}(A)$  by  $\mathcal{WE}(A)$ .

**Corollary 9.2.** *Let  $\mathcal{X}$  be a reflexive Banach space,  $A$  an operator on  $\mathcal{X}$ . Then  $\{R^* : R \in \mathcal{WE}(A)\} = WG(K)$ , where  $K = A^*(B_{\mathcal{X}^*})$ .*

*Proof.* Since  $\mathcal{X}$  is reflexive the map  $R \rightarrow R^*$  from  $\mathcal{L}(\mathcal{X})$  to  $\mathcal{L}(\mathcal{X}^*)$  is bicontinuous in the WOT-topologies. Hence the result follows from Lemma 9.1.  $\square$

**Corollary 9.3.** *Let  $\mathcal{X}$  be reflexive. If  $A \in \mathcal{L}(\mathcal{X})$  is such that the sequence*

$$\{d_n(A^*(B_{\mathcal{X}^*}))\}_{n=0}^\infty$$

*is lacunary, then  $\mathcal{WE}(A)$  is contained in the set of all operators for which  $\ker A$  is an invariant subspace.*

*Proof.* Follows immediately from Lemma 3.5 if we take into account the observation that  $\ker A$  is an invariant subspace of  $R$  if and only if  $\overline{A^*\mathcal{X}^*}$  is an invariant subspace of  $R^*$  (that is, if and only if  $R^* \in \mathcal{A}_K$ ).  $\square$

Applying Theorem 2.1, we obtain the converse inclusion:

**Corollary 9.4.** *The set of all operators preserving  $\ker A$  is contained in  $\mathcal{WE}(A)$ . If  $\ker A = \{0\}$ , then  $\mathcal{WE}(A) = \mathcal{L}(\mathcal{X})$ .*

Applying Theorem 4.6, we get

**Corollary 9.5.** *If  $\mathcal{X}$  is a separable Hilbert space and  $A \in \mathcal{L}(\mathcal{X})$  is such that the sequence  $s$ -numbers of  $A$  is not lacunary, then  $\mathcal{WE}(A) = \mathcal{L}(\mathcal{X})$ .*

We can summarize Hilbert-space-case results in the following way:

**Theorem 9.6** (A complete classification in the Hilbert space case). *Let  $\mathcal{X}$  be a separable Hilbert space.*

- (i) *If the sequence of  $s$ -numbers of  $A$  is not lacunary, then  $\mathcal{WE}(A) = \mathcal{L}(\mathcal{X})$ .*
- (ii) *If the sequence of  $s$ -numbers of  $A$  is lacunary, then  $\mathcal{WE}(A)$  coincides with the set of operators for which  $\ker A$  is an invariant subspace.*

Finally, using Theorem 4.10 we obtain a result on the ultra-weak closure of  $\mathcal{WE}(A)$ :

**Corollary 9.7.** *Let  $A \in \mathcal{L}(\mathcal{H})$  be such that its sequence of  $s$ -numbers is not lacunary. Then the closure of the set (19) in the ultra-weak topology coincides with  $\mathcal{L}(\mathcal{H})$ .*

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